Sequential Locally Optimum Test (SLOT): A Sequential Detection Scheme Based on Locally Optimum Test Statistic

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SUMMARY Based on the characteristics of the thresholds of two detection schemes employing locally optimum test statistics, a sequential detection design procedure is proposed and analyzed. The proposed sequential test, called the sequential locally optimum test (SLOT), inherently provides finite stopping time (terminates with probability one within the finite horizon), and thereby avoids undesirable forced termination. The performance of the SLOT is compared with that of the fixed sample-size test, sequential probability ratio test (SPRT), truncated SPRT, and 2-SPRT. It is observed that the SLOT requires smaller average sample numbers than other schemes at most values of the normalized signal amplitude while maintaining the error performance close to the SPRT.

key words: asymptotic sample number, locally optimum detector, minimum false-alarm, sequential test, sequential probability ratio test

1. Introduction

A locally optimum (LO) detector is known to maximize the (generalized) slope of the power function as the signal-to-noise ratio approaches zero [1], [2]. The class of LO detectors is therefore quite useful for the detection of signals when the strength of a signal is weaker than that of noise processes. In addition, an LO detector generally offers the advantage of simpler structures and practical implementations than the uniformly most powerful (UMP) or optimum detectors.

Because of such advantages, the LO detectors have been considered in a variety of environments [3]–[8]. Specifically, the LO detection in a generalized observation model is addressed in [3] and a robust LO detector for signals embedded in additive non-Gaussian noise is studied in [4]. In [5], locally most powerful and invariant tests are combined into a new detector to test the presence of a signal in additive noise. As an application of the LO detector in direct sequence spread spectrum systems, code acquisition methods based on local optimality have been studied in [6]. The method of LO detection has also been applied in such diverse areas as multiplicative watermarks [7]. Recently in [8], Markov noise has been taken into consideration for the LO detection of signals in multiplicative noise.

Among the abundance of sequential schemes [9]–[14], due to the Wald-Woldowitz theorem [15], the sequential probability ratio test (SPRT) is known to provide a significant advantage in the size (a measure of complexity or efficiency) of sample required to satisfy given constraints on the false-alarm and miss-detection probabilities [16]–[18]. Specifically, the SPRT has a smaller average sample number (ASN) than any other sequential test and the fixed sample-size test (FSST) when the actual value of the signal amplitude (strength) is equal to the putative value assumed at the time of design. The asymptotic behavior of the relative efficiency of the SPRT to the FSST has been studied in [19] for the detection of a constant signal in additive noise. When the constraints of the false-alarm and miss-detection probabilities are very low, however, the ASN of the SPRT could become very large: a mixture of the SPRT and FSST, called the truncated SPRT (TSPT) [20], and a combination of two SPRTs (2-SPRT) [9] as a solution to the Kiefer-Weiss problem [11]–[13], [21] of minimizing the ASN have been investigated to overcome such a drawback of the SPRT.

Other modified versions with simplicity in structure and analysis have also been proposed and applied to discrete-time signal detection as in [22]. In [23], sequential detection of weak signals using Taylor series approximation of the likelihood ratio has been considered. Multi-hypothesis techniques of the binary SPRT are proposed and analyzed in [24], [25], where the asymptotic approximations of the ASN and asymptotic optimality in the sense of minimizing the ASN are addressed. In addition, there exist several other approaches to optimizing sequential tests when the hypotheses are composite. In [13], for example, exponential families have been considered in the context of multiple composite hypotheses. More specifically, adaptive multi-hypothesis versions of the Robbins-Siegmund one-sided tests have been analyzed: the uniform asymptotic optimality of these tests in the first-order has been proved under certain conditions, and a connection with the Kiefer-Weiss problem has also been discussed.

Interestingly, the sequential detection problem has also been studied in bio-statistics, where a certain decision on clinical trials should be made as early as possible because additional trials in most cases accompany costly processes and could imply an ethical issue: in fact, the concept of the overall false-alarm and miss-detection probabilities are introduced in [26]. Later, a detailed design with the concept of
error spending function and per-observation false-alarm and miss-detection probabilities has been introduced in [27].

With these backgrounds, we propose in this paper the design procedure of a detection scheme as a combination of the sequential detection criterion and LO test statistics. The design procedure is based on the equal error spending function, where the false-alarm probabilities at the stages of the sequential observation are equal and their sum is less than or equal to the constraints on the overall false-alarm probability. The same principle is applied also to the miss-detection probability.

The rest of this paper is organized as follows. In Sect. 2, a discrete-time observation model is described, and detection schemes employing the LO test statistics are addressed as preliminaries. Some characteristics of the thresholds are discussed in Sect. 3. The proposed sequential detection design procedure is then described in Sect. 4. We investigate the performance of the proposed sequential test against other schemes through computer simulations in Sect. 5.

2. System Model and Test Statistics

2.1 Observation Model

Let us consider a model in which the observation \( z_m \) = \( [z_1, z_2, \cdots, z_m]^T \) is described by

\[
\tilde{z}_m = \theta s_m + v_m,
\]

where \( m \) denotes the sample size of an FSST scheme, \( \theta > 0 \) is the signal strength approaching zero, and \( s_m = [s_1, s_2, \cdots, s_m]^T \) is the known signal vector of finite non-negative elements \( [s_i]_{i=1}^m \). The additive noise vector \( v_m = [v_1, v_2, \cdots, v_m]^T \) is assumed to be an independent and identically distributed (i.i.d.) random vector whose joint probability density function (pdf) is

\[
f_{v_m}(v_m) = \prod_{i=1}^m f_v(v_i)
\]

with \( f_v \) the common marginal pdf of \( v_i \) such that \( f_v(v_i) = f_v(-v_i) \), where \( v_m = [v_1, v_2, \cdots, v_m]^T \) is an \( m \)-dimensional real vector. Here, without loss of generality, we assume

\[
\lim_{m \to \infty} p_s(m) = P_s
\]

for a non-zero finite value \( P_s \), where

\[
p_s = \frac{1}{m} \sum_{i=1}^m s_i^2.
\]

Under the observation model of Eq. (1), it is straightforward to express the problem of signal detection by a statistical hypothesis testing problem. Denoting the null and alternative hypotheses by \( H_0 \) and \( H_1 \), respectively, we have

\[
H_0 : \quad \tilde{z}_m = v_m,
\]

\[
H_1 : \quad \tilde{z}_m = \theta s_m + v_m.
\]

In this binary decision problem, the \( m \)-dimensional real space \( \mathbb{R}^m \) of the random vector \( z_m \) is partitioned into two decision regions \( \mathbb{R}^m_{L} \) for decision \( D_0 \) and \( \mathbb{R}^m_{L} \) for decision \( D_1 \), where \( D_0 \) and \( D_1 \) denote the decisions accepting hypotheses \( H_0 \) and \( H_1 \), respectively. Note that we have

\[
f_{z_m}(z_m|H_0) = \frac{f_{z_m}(z_m|H_1)|_{\theta=0}}{\prod_{i=1}^m f_v(v_i)}
\]

where \( f_{z_m}(z_m|H_0) \) and \( f_{z_m}(z_m|H_1) \) are the pdf’s of the observation vector \( z_m \) under the null and alternative hypotheses, respectively, and \( z_m = (z_1, z_2, \cdots, z_m) \) is a vector in \( \mathbb{R}^m \).

2.2 The LO and Minimum False-Alarm Detection Schemes

Based on the generalized Neyman Pearson lemma [28], the test function of the LO detector can be expressed as

\[
\delta_{LO}(z_m) = \begin{cases} 1, & \Lambda_{LO}(z_m) \geq \tau_{LO,\alpha}(m), \\ 0, & \Lambda_{LO}(z_m) < \tau_{LO,\alpha}(m), \end{cases}
\]

where

\[
\Lambda_{LO}(z_m) = \frac{f_{z_m}^{(\nu)}(z_m|H_1)|_{\theta=0}}{f_{z_m}^{(\nu)}(z_m|H_0)}
\]

is called the LO test statistic with \( \nu \) the first non-zero derivative of \( f_{z_m}^{(\nu)}(z_m|H_1) \) at \( \theta = 0 \). The threshold \( \tau_{LO,\alpha}(m) \) in Eq. (7) is determined to satisfy the constraint that the false-alarm probability \( \text{Pr}[D_1|H_0] \) is no higher than the pre-assigned constant \( \alpha \). The LO test function \( \delta_{LO}(\cdot) \) is known [1], [28] to maximize the (generalized) slope

\[
\int_{\mathbb{R}^m} \int_{\mathbb{R}^m} d\bar{z}_m
\]

of the power function at \( \theta = 0 \) among all the test functions \( \delta(\cdot) \) satisfying the same constraint on the false-alarm probability \( \text{Pr}[D_1|H_0] \).

It is straightforward to show that \( \nu = 1 \) for the detection problem in this paper, and thus the LO test statistic can be obtained as

\[
\Lambda_{LO}(z_m) = \sum_{i=1}^m s_i g_{LO}(z_i)
\]

from Eq. (8), where

\[
g_{LO}(x) = \frac{f_v(x)}{f_v(x)}
\]

is called the LO nonlinearity.

Now, recall that we have
\[
\text{Pr}[D_0|H_0] + \text{Pr}[D_1|H_0] = 1 \quad (12)
\]

and

\[
\text{Pr}[D_0|H_1] + \text{Pr}[D_1|H_1] = 1 \quad (13)
\]
in the binary decision situation, where the false-alarm probability \( \text{Pr}[D_1|H_0] \) and miss-detection probability \( \text{Pr}[D_0|H_1] \) are the error probabilities, and \( \text{Pr}[D_1|H_1] \) is the detection probability.

On the analogy of the LO detection, let us derive a detection scheme minimizing the false-alarm probability \( \text{Pr}[D_1|H_0] \) under the constraint that the miss-detection probability \( \text{Pr}[D_0|H_1] \) is no higher than a pre-assigned number \( \beta_m \). Expanding \( f_{z_m} (z_m|H_1) \) by Taylor series about \( \theta = 0 \) and approximating the result with Eq. (6), we have

\[
f_{z_m} (z_m|H_1) = f_{z_m} (z_m|H_1)\bigg|_{\theta=0} + \sum_{i=1}^{\infty} \frac{\theta^i}{i!} f_{z_m}^{(i)} (z_m|H_1)\bigg|_{\theta=0}
\approx f_{z_m} (z_m|H_0) + \frac{\theta}{\nu^1} f_{z_m}^{\nu^1} (z_m|H_1)\bigg|_{\theta=0}. \quad (14)
\]
The miss-detection probability can then be written as

\[
\text{Pr} \{D_0|H_1\} = \int_{\mathbb{R}^m} f_{z_m} (z_m|H_0) d\tilde{z}_m
\approx \int_{\mathbb{R}^m} \left\{ f_{z_m} (z_m|H_0) + \frac{\theta}{\nu^1} f_{z_m}^{\nu^1} (z_m|H_1) \bigg|_{\theta=0} \right\} d\tilde{z}_m. \quad (15)
\]

Now, to minimize the false-alarm probability \( \text{Pr}[D_1|H_0] \) under the constraint \( \text{Pr}[D_0|H_1] \leq \beta_m \), let us employ Lagrange functional and consider

\[
\tilde{\Gamma}_{1,m} = \text{Pr}[D_1|H_0] - \lambda_{1,m} [\text{Pr}[D_0|H_1] - \beta_m] \\
\approx 1 - \text{Pr}[D_0|H_0] + \lambda_{1,m} \beta_m - \lambda_{1,m} \\
\cdot \int_{\mathbb{R}^m} \left\{ f_{z_m} (z_m|H_0) + \frac{\theta}{\nu^1} f_{z_m}^{\nu^1} (z_m|H_1) \bigg|_{\theta=0} \right\} d\tilde{z}_m
\]

\[
= 1 + \lambda_{1,m} \beta_m - \int_{\mathbb{R}^m} \left\{ (1 + \lambda_{1,m})f_{z_m} (z_m|H_0) + \lambda_{1,m} \frac{\theta}{\nu^1} f_{z_m}^{\nu^1} (z_m|H_1) \bigg|_{\theta=0} \right\} d\tilde{z}_m, \quad (16)
\]

where \( \lambda_{1,m} \) is the Lagrange multiplier and we have used Eq. (15). Based on Eq. (16), a detection scheme, called the minimum false-alarm (MF) scheme in this paper, can be designed. Specifically, first note that only the integrand needs to be maximized in order to minimize the quantity \( \tilde{\Gamma}_{1,m} \) in Eq. (16) since \( \lambda_{1,m} \beta_m \) is a constant. The minimization of \( \tilde{\Gamma}_{1,m} \) (or equivalently, the maximization of the integrand) can be achieved if the values of \( z_m \) for which the integrand in Eq. (16) is nonnegative are associated with the decision region \( \mathbb{Z}_{0,m} \). Therefore, considering that the \( m \)-dimensional real space \( \mathbb{Z}_m \) is partitioned into the two decision regions \( \mathbb{Z}_{0,m} \) and \( \mathbb{Z}_{1,m} \), the test function \( \delta_{MF}(z_m) \) which minimizes Eq. (16) can be expressed as

\[
\delta_{MF}(z_m) = \begin{cases} 1, & \Lambda_{LO}(z_m) \geq \tau_{MF,\beta_m}(m), \\
0, & \Lambda_{LO}(z_m) < \tau_{MF,\beta_m}(m). \end{cases} \quad (17)
\]

In practice, the threshold \( \tau_{MF,\beta_m}(m) = \frac{(1+\lambda_{1,m})\nu^1}{\theta \nu^1} \) in Eq. (17) is determined from the condition \( \text{Pr}[D_0|H_1] = \beta_m \).

Note that the test statistic \( \Lambda_{LO}(z_m) \) used in Eq. (17) is the LO test statistic used also in Eq. (7): it should nonetheless be noted that the two thresholds \( \tau_{LO,\alpha_m}(m) \) in Eq. (7) and \( \tau_{MF,\beta_m}(m) \) in Eq. (17) are different in general. In addition, when \( \tau_{LO,\alpha_m}(m) > \tau_{MF,\beta_m}(m) \), it is not possible to make a decision with \( m \) observations in such a way that the constraints \( \alpha_m \) on the false-alarm probability and \( \beta_m \) on the miss-detection probability are satisfied simultaneously. In this case, additional observations can help the decision, which leads to the sequential detection problem.

3. Analysis of Threshold

3.1 Asymptotic Distributions

Assuming \( \alpha_1 = \alpha_2 = \cdots = \alpha_F \) and \( \beta_1 = \beta_2 = \cdots = \beta_F \) for some constants \( \alpha_F \) and \( \beta_F \), let us explore some characteristics of the thresholds \( \tau_{LO,\alpha_m}(m) \) and \( \tau_{MF,\beta_m}(m) \) as the sample size \( m \) varies. As is well-known in the theory of local asymptotic normality \[29, 30\], the distribution of \( \{\Lambda_{LO}(z_m) - \mu_A\}/\sigma_A \) approaches asymptotically the standard normal distribution from the central limit theorem when \( m \to \infty \) since the random variables \( \{g_{LO}(z_i)|m\}_{i=1}^m \) are independent and their means and variances are finite: this has been specifically established in \[1\]. Here, \( \mu_A \) and \( \sigma_A^2 \) are the mean and variance of \( \Lambda_{LO}(z_m) \), respectively, and depend on the hypotheses. More specifically, under the null hypothesis \( H_{0,m} \), we have

\[
\mu_A = \mu_{0,m} = 0 \quad (18)
\]

since we can assume \( \lim_{x \to -\infty} f_\theta(x) = 0 \) almost always, and

\[
\sigma_A^2 = \sigma_{0,m}^2 = mI(f_\theta)p_\theta(m), \quad (19)
\]

where

\[
I(f_\theta) = \int_{-\infty}^{\infty} g_{LO}(x)f_\theta(x)dx
\]

\[
= -\int_{-\infty}^{\infty} g_{LO}(x)f'_\theta(x)dx \quad (20)
\]
is the Fisher information \[2\] of \( f_\theta \) and we have used additional subscript \( m \) to specify the sample size (for instance, in \( H_{0,m} \) and \( \mu_{0,m} \)). Similarly, we have

\[
\mu_A = \mu_{1,m} = \sum_{i=1}^{m} S_iE[g_{LO}(\theta S_i + \nu_i)]
\]

\[
\approx \theta mI(f_\theta)p_\theta(m) \quad (21)
\]

using Eq. (4) and \( f_\theta(z_i - \theta_S) \approx f_\theta(z_i) - \theta_S f'_\theta(z_i) \) for \( \theta \approx 0 \),
and
\[ \sigma_{0,m}^2 = \sigma_{0,m}^2 \approx \sigma_{0,m}^2 \tag{22} \]
under the alternative hypothesis \(H_{1,m} \). More details of the derivation for the approximations in Eqs. (21) and (22) are given in Appendix. Clearly, when \( \theta = 0 \) or when the noise \( \mathbf{p}_m \) is a zero-mean Gaussian vector, we have equalities instead of approximations in Eqs. (21) and (22).

Now, when the false alarm probability \( \Pr(D_{1,m}|H_{0,m}) \) is equal to the constraint \( \alpha_F \), we have
\[ \alpha_F = Q \left( \frac{1}{\sigma_{0,m}} \right) \tag{23} \]
using Eqs. (18) and (19), and consequently,
\[ \tau_{LO,\alpha}(m) \approx \sigma_{0,m} Q^{-1}(\alpha_F), \tag{24} \]
where \( Q^{-1}(x) \) is the inverse of the complementary cumulative distribution function (ccdf), or tail probability,
\[ Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} \exp \left\{ -\frac{t^2}{2} \right\} dt \tag{25} \]
of the standard normal distribution \( N(0,1) \). In a similar manner, we can obtain
\[ \beta_F \approx 1 - Q \left( \frac{1}{\sigma_{1,m}} \right) \tag{26} \]
and therefore,
\[ \tau_{MF,\beta}(m) \approx \mu_{1,m} + \sigma_{1,m} Q^{-1}(1 - \beta_F) \tag{27} \]
when the miss-detection probability \( \Pr(D_{0,m}|H_{1,m}) \) is equal to the constraint \( \beta_F \). More details of the derivation for the approximation of Eq. (27) are given in Appendix. It is again noteworthy that, for a zero-mean Gaussian noise \( \mathbf{p}_m \) in the observation model of Eq. (1), the approximations in Eqs. (21)–(24), (26), and (27) will become equalities.

3.2 The Thresholds

Let us now characterize the behavior of the thresholds \( \tau_{LO,\alpha}(m) \) and \( \tau_{MF,\beta}(m) \) shown in Eqs. (24) and (27) as functions of the sample size \( m \). First, we have
\[ \tau_{LO,\alpha}(m+1) - \tau_{LO,\alpha}(m) \approx (\sigma_{0,m+1} - \sigma_{0,m}) Q^{-1}(\alpha_F) = \left( \sqrt{m+1} - \sqrt{m} \right) \sqrt{I(f_0)p_s(m+1)} Q^{-1}(\alpha_F) \approx \left( \sqrt{m+1} - \sqrt{m} \right) \sqrt{I(f_0)p_s} Q^{-1}(\alpha_F) \tag{28} \]
for a sufficiently large value of \( m \) from Eq. (24) using Eqs. (3), (4), and (19). The first and second approximations in Eq. (28) can be replaced with an equality when the noise is zero-mean Gaussian and when \( s_i \) does not depend on \( i \), respectively. Since the constant \( \alpha_F \) is normally close to zero, we have \( Q^{-1}(\alpha_F) > 0 \). In addition, the positive quantity \( \sqrt{m+1} - \sqrt{m} \) becomes smaller as \( m \) increases. Thus, it is clear from Eq. (28) that \( \tau_{LO,\alpha}(m) \) increases and its rate of increase decreases (to zero eventually) as \( m \) increases.

Similarly, we have
\[ \tau_{MF,\beta}(m+1) - \tau_{MF,\beta}(m) \approx \theta I(f_0)P_s + \left( \sqrt{m+1} - \sqrt{m} \right) \sqrt{I(f_0)p_s} Q^{-1}(1 - \beta_F) \tag{29} \]
from Eq. (27) using Eqs. (3), (4), (21), and (22) assuming \( \mu_{1,m} \approx P_s \) when \( \theta \) is sufficiently small and \( m \) is sufficiently large. Noting that \( Q^{-1}(1 - \beta_F) \) is normally close to zero, it can be shown after some manipulations using Eq. (29) that \( \tau_{MF,\beta}(m) \) becomes smaller and larger as \( m \) increases when \( m < m'(\beta_F) \) and \( m > m'(\beta_F) \), respectively, and that the rate of increase of \( \tau_{MF,\beta}(m) \) grows higher as \( m \) increases when \( m > m'(\beta_F) \), where
\[ m'(\beta_F) = \frac{1}{4} \left\{ \frac{\theta I(f_0)P_s}{Q^{-1}(1 - \beta_F) - \theta I(f_0)p_s} \right\}^2. \tag{30} \]

Next, consider \( \tau_{LO,\alpha}(1) \) with \( \tau_{MF,\beta}(1) \). When both \( \alpha_F \) and \( \beta_F \) are small enough with \( \theta \) approaching zero, it is straightforward to see
\[ \tau_{LO,\alpha}(1) > \tau_{MF,\beta}(1) \tag{31} \]
since \( \tau_{LO,\alpha}(1) > 0 \) and \( \tau_{MF,\beta}(1) < 0 \) from
\[ \alpha_F = \int_{\tau_{LO,\alpha}(1)}^{\tau_{MF,\beta}(1)} f_{\Lambda_1}(\lambda) |H_{0,1}| d\lambda \tag{32} \]
and
\[ \beta_F = \int_{-\infty}^{\tau_{MF,\beta}(1)} f_{\Lambda_1}(\lambda) |H_{1,1}| d\lambda \approx \int_{-\infty}^{\tau_{MF,\beta}(1)} f_{\Lambda_1}(\lambda) |H_{1,1}| d\lambda \tag{33} \]
using Eq. (6). In Eqs. (32) and (33), \( f_{\Lambda_1}(\lambda)|H_{0,1}| \) and \( f_{\Lambda_1}(\lambda)|H_{1,1}| \) are the pdf’s of the random variable \( \Lambda_1 = \Lambda_{LO}(Z_1) \) under the null and alternative hypotheses, respectively.

We have already observed in Eq. (28) that \( \tau_{LO,\alpha}(m) \) grows larger with the rate of increase becoming lower as \( m \) increases. This fact and Eq. (31), together with that \( \tau_{MF,\beta}(m) \) grows larger with the rate of increase becoming higher as \( m \) when \( m > m'(\beta_F) \), imply that \( \tau_{MF,\beta}(m) \) will eventually be larger than \( \tau_{LO,\alpha}(m) \). In other words, we have
\[ \tau_{LO,\alpha}(m) > \tau_{MF,\beta}(m), \]
for \( m = 1, 2, \ldots, m'(\alpha_F, \beta_F) - 1 \),
\[ \tau_{LO,\alpha}(m) \leq \tau_{MF,\beta}(m), \]
for \( m = m'(\alpha_F, \beta_F), m'(\alpha_F, \beta_F) + 1, \ldots \)
where
\[ m'(\alpha_F, \beta_F) = \arg \min_m \left\{ \tau_{LO,\alpha}(m) \leq \tau_{MF,\beta}(m) \right\} \approx \arg \min_m \left\{ \sqrt{mI(f_0)p_s} Q^{-1}(\alpha_F) \right\} \leq \]
Theorem 35: Let \( \{ s_i \}_{i=1}^m \) be an additive white Gaussian noise (AWGN) vector with mean 0 and variance 1, \( \theta = 0.5 \), and \( \alpha_F, \beta_F = 10^{-3} \). Then, we have \( m^* = 9.06 \) from Eq. (30), implying that \( \tau_{MF,\beta}(m) \) decreases until \( m \geq 9 \) and then increases when \( m \geq 10 \) as the sample size \( m \) increases. In addition, we have \( m^*(\alpha_F, \beta_F) = 153 \) from Eq. (35). As other examples, assume that all the elements of signal vector are again 1, \( \theta = 1 \), and \( \alpha_F = 10^{-3} \) when the pdf \( f_{\beta} \) is Cauchy pdf

\[
f_c(\chi) = \frac{1}{\pi(x^2 + 1)}
\]

and logistic pdf

\[
f_L(\chi) = \frac{\pi \exp \left( \frac{\chi}{\sqrt{3}} \right)}{\sqrt{3} \left( 1 + \exp \left( \frac{\chi}{\sqrt{3}} \right) \right)^2}.
\]

Figures 1 and 2 show the thresholds \( \tau_{LO,\alpha}(m) \) and \( \tau_{MF,\beta}(m) \) as functions of the sample size \( m \) in the Cauchy and logistic cases, respectively, obtained from 10⁴ Monte Carlo runs at each value of \( m \). It is observed that the thresholds \( \tau_{LO,\alpha}(m) \) and \( \tau_{MF,\beta}(m) \) in the Cauchy and logistic noise environment exhibit a tendency similar to that in the AWGN environment. This indirectly and partially supports that the threshold characteristics derived herein hold in some of the non-Gaussian noise environment also.

4. Design of Sequential Detection

We have addressed the sequences \( \{ \tau_{LO,\alpha}(m) \}_{m=1}^{\infty} \) and \( \{ \tau_{MF,\beta}(m) \}_{m=1}^{\infty} \) of thresholds obtained from the LO-based FSST schemes \( \delta_{LO}(Z_m) \) and \( \delta_{MF}(Z_m) \), respectively; it is observed that the two sequences of thresholds ‘intersect’ at some value of \( m \), allowing possibly useful application in sequential detection problems. This observation is the basis/motivation on which a sequential detection scheme is designed in the next section.

4.1 The Proposed Scheme

In sequential detection, the \( m \)-dimensional observation space \( Z_m \) is partitioned into three regions \( [Z_m]_{i=0}^2 \) when \( m \) observations are available, where \( Z_{0,m} \) and \( Z_{1,m} \) are the same as in the binary decision problem based on Eq. (5) and \( Z_{2,m} \) denotes the region for postponing the decision. Define by \( D_{i,m} \) the event (action) that decision \( i \) is made at the \( m \)-th
observation, where decisions 0, 1, and 2 denote accepting $H_0$, accepting $H_1$, and deferring the binary decision, respectively.

When a sequential detection scheme is employed, signals in noise can be detected by augmenting additional observations at each stage. For each additional observation, the sequential detection can make one of the three decisions: consequently, the size of the observations required for choosing $H_0$ or $H_1$ is not pre-determined. A noticeable characteristic of the sequential detection is that, on the average, it requires substantially smaller observation size than an equally reliable FSST does with fixed observation size.

The goal of sequential detection is to detect the signal as fast as possible, or equivalently, to minimize the ASN while satisfying prescribed constraints on the false-alarm and miss-detection probabilities. Here, writing the ASN as a function of $\theta$ explicitly, we have

$$\text{ASN}_\theta = \sum_{m=1}^{M} m \cdot \Pr \left( \text{Stage } m \text{ is the first stage accepting } H_0 \text{ or } H_1 \mid \theta \right)$$

$$= \Pr \left( D_{0,1} \cup D_{1,1} \mid \theta \right) + \sum_{m=2}^{M} m \cdot \Pr \left( D_{0,m} \cup D_{1,m} \right) \setminus \left( \bigcap_{j=1}^{m-1} D_{2,j} \right) \mid H_0 \right)$$

$$\cdot \Pr \left( D_{1,m} \setminus \left( \bigcap_{j=1}^{m-1} D_{2,j} \right) \mid H_0 \right) + \sum_{m=2}^{M} m \cdot \Pr \left( D_{0,m} \cap \left( \bigcap_{j=1}^{m-1} D_{2,j} \right) \mid H_0 \right)$$

where the ‘total’ false-alarm and miss-detection probabilities are

$$P_{FA} = \Pr \left( D_{1,1} \mid H_0 \right)$$

$$+ \sum_{m=2}^{M} \Pr \left( D_{1,m} \cap \left( \bigcap_{j=1}^{m-1} D_{2,j} \right) \mid H_0 \right)$$

and

$$P_{MD} = \Pr \left( D_{0,1} \mid H_1 \right)$$

$$+ \sum_{m=2}^{M} \Pr \left( D_{0,m} \cap \left( \bigcap_{j=1}^{m-1} D_{2,j} \right) \mid H_1 \right)$$

respectively. In Eqs. (38)–(40), the parameter $M$ denotes the stage at which the sequential scheme terminates (that is, makes a final decision).

In the minimization of the ASN depicted as Eq. (38), a sequential scheme is required to satisfy

$$P_{FA} \leq \alpha \quad \text{and} \quad P_{MD} \leq \beta$$

where the constants $\alpha$ and $\beta$ are pre-determined constraints on the error probabilities.

Now, utilizing the LO test statistic of Eq. (8) in the sequential decision procedure, the decision rule at stage $m$ can be expressed as

Accept $H_0$ and stop if $\Lambda_{LO}(z_m) \leq \tau_m^L$.
Accept $H_1$ and stop if $\Lambda_{LO}(z_m) > \tau_m^U$.
Defer the decision if $\tau_m^L < \Lambda_{LO}(z_m) \leq \tau_m^U$.  \hspace{1cm} (42)

The sequential scheme proposed by the decision rule of Eq. (42) will be called the sequential LO test (SLOT).

The same statistic $\Lambda_{LO}(z_m)$ is used in the FSST shown in Sect. 2 and thus the thresholds of the SLOT is somehow related to those of the LO-based FSST. Nonetheless, the thresholds $[\tau_{m}^L]_{m=1}^{M}$ and $[\tau_{m}^U]_{m=1}^{M}$ of the SLOT should clearly be set (obtained) differently from the thresholds $[\tau_{LO,m}(m)]_{m=1}^{\infty}$ and $[\tau_{MF,m}(m)]_{m=1}^{\infty}$ described in Eqs. (7) and (17). Specifically, in the FSST where the number of observations is fixed and there exists only one threshold in the decision function, the threshold is determined from the total error probabilities and the number of observations for a single decision. On the other hand, as the error probabilities in a sequential test are in general distributed over the stages until the final decision is made under the condition that the sums of the per-stage error probabilities satisfy the constraints of the total false-alarm and miss-detection probabilities, the thresholds are determined based on the per-stage error probabilities.

4.2 Design Procedure: Termination Stage $M$ and Thresholds $[\tau_m^L]$ and $[\tau_m^U]$ 

As described briefly in the previous section, per-stage error probabilities should be available in the design process. Clearly, there exist a number of possibilities in ‘distributing’ the constraints $\alpha$ and $\beta$ on the total error probabilities over the $M$ stages in sequential detection schemes; in this paper, we adopt the ‘equal error spending’ method in the SLOT mainly because of its tractability and simplicity in the analysis. Specifically, we will distribute the requirement $\alpha$ on the overall false-alarm probability over every stage evenly as

$$\alpha_m = \frac{\alpha}{M}, \quad m = 1, 2, \cdots, M.$$  \hspace{1cm} (43)

In a similar manner, the requirement $\beta$ on the overall miss-detection probability is distributed over the stages evenly as

$$\beta_m = \frac{\beta}{M}, \quad m = 1, 2, \cdots, M.$$  \hspace{1cm} (44)

4.2.1 Constraints and Thresholds 

Now, assume that we have somehow obtained $M$ already. Then, using the per-stage decision function shown in Eq. (42), we have

$$\Pr \left( \bigcap_{m=1}^{M-1} R_m(z_m) \Lambda_{LO}(z_m) \geq \tau_m^U \mid H_0 \right) = \Pr \left( \bigcap_{m=1}^{M-2} R_m(z_m), \Lambda_{LO}(z_{M-1}) \geq \tau_{M-1}^U \mid H_0 \right)$$

$$= \Pr \left( R_1(z_1), \Lambda_{LO}(z_{M-1}) \geq \tau_{M-1}^U \mid H_0 \right)$$

$$= \Pr \left( \Lambda_{LO}(z_M) \geq \tau_M^U \mid H_0 \right) = \frac{\alpha}{M}.$$  \hspace{1cm} (45)
by employing the per-stage error probability of Eq. (43), where \( R_m(z_m) \) denotes the event

\[
R_m(z_m) = \{ \tau_m^U < \Lambda_{LO}(z_m) < \tau_m^L \}
\]

(46)

for \( m = 1, 2, \cdots, M - 1 \). Similarly, for the miss-detection probabilities, we have

\[
\Pr \left( \bigcap_{m=1}^{M-1} R_m(z_m), \Lambda_{LO}(z_M) \leq \tau_M^L \mid H_1 \right) = \Pr \left( \bigcap_{m=1}^{M-2} R_m(z_m), \Lambda_{LO}(z_{M-1}) \leq \tau_{M-1}^L \mid H_1 \right) \]

\[
\vdots
\]

\[
= \Pr \left( R_1(z), \Lambda_{LO}(z) \leq \tau_1^L \mid H_1 \right) = \frac{\beta}{M}
\]

(47)

by employing the per-stage error probability of Eq. (44) in the per-stage decision function of Eq. (42).

The sets \( \{ \tau_{m1}^U \}_{m=1}^M \) and \( \{ \tau_{m1}^L \}_{m=1}^M \) of 2M thresholds of the SLOT can now be determined from the 2M equations given by Eqs. (45) and (47) in a step-by-step manner. Specifically, \( \tau_1^U \) and \( \tau_1^L \) are first determined from

\[
\int_{\Lambda_{LO}(z) \geq \tau_1^U} f_{z_1}(z_1 | H_0)dz_1 = \frac{\alpha}{M}
\]

(48)

and

\[
\int_{\Lambda_{LO}(z) \leq \tau_1^L} f_{z_1}(z_1 | H_1)dz_1 = \frac{\beta}{M}
\]

(49)

using Eqs. (45) and (47), respectively. Then, the thresholds \( \tau_m^U \) and \( \tau_m^L \) for \( m = 2, 3, \cdots, M \) are determined from

\[
\int_{R(z)} \int_{\Lambda_{LO}(z) \geq \tau_m^U} f_{z_m}(z_m | H_0)dz_m = \frac{\alpha}{M}
\]

(50)

and

\[
\int_{R(z)} \int_{\Lambda_{LO}(z) \leq \tau_m^L} f_{z_m}(z_m | H_1)dz_m = \frac{\beta}{M}
\]

(51)

recursively using \( \{ \tau_{i1}^U \}_{i=1}^{m-1} \) and \( \{ \tau_{i1}^L \}_{i=1}^{m-1} \) obtained already.

4.2.2 Iterative Determination

To determine the thresholds \( \{ \tau_{m1}^U \}_{m=1}^M \) and \( \{ \tau_{m1}^L \}_{m=1}^M \) in the SLOT, we start with an initial value of \( M \) chosen arbitrarily, and then repeat iteration by changing the value of \( M \) until the two sets \( \{ \tau_{m1}^U \}_{m=1}^M \) and \( \{ \tau_{m1}^L \}_{m=1}^M \) of thresholds possess the property \( \tau_M^U \leq \tau_M^L \).

Specifically, given the constraints \( \alpha \) and \( \beta \) on the total false-alarm and miss-detection probabilities, select a value of \( M \) arbitrarily and compute \( \{ \tau_{m1}^U \}_{m=1}^M \) and \( \{ \tau_{m1}^L \}_{m=1}^M \) from Eqs. (48)–(51). During the calculation of the thresholds, (1) if there exists \( m (< M) \) such that \( \tau_m^U < \tau_m^L \), then decrease \( M \) by one, (2) if \( \tau_m^U = \tau_m^L \), stop the iteration since it means we have obtained the desired parameters, or (3) if \( \tau_M^U > \tau_M^L \), increase \( M \) by one. As necessary, the thresholds \( \{ \tau_{m1}^U \}_{m=1}^M \) and \( \{ \tau_{m1}^L \}_{m=1}^M \) are re-computed with the value of \( M \) updated. The procedure continues until \( \tau_M^U \leq \tau_M^L \). As there is usually no closed-form solution for Eqs. (48)–(51) unfortunately, the repetition procedure relying on numerical evaluation is unavoidable.

Figure 3 shows examples of thresholds for various values of \( \alpha \) and \( \beta \), where it is assumed that the noise is zero-mean unit-variance AWGN and the signal strength \( \theta = 1 \).

In passing, it should be mentioned that the determination of the thresholds \( \{ \tau_{m1}^U \}_{m=1}^M \) and \( \{ \tau_{m1}^L \}_{m=1}^M \) could require a non-negligible computation in some cases: yet, the thresholds can be pre-determined in an off-line fashion before a specific application. Once the thresholds are so determined, they can be employed readily in the SLOT for on-line (real-time) processing.

5. Complexity Comparisons

5.1 Comparison Environments

The performance of the SLOT, proposed and designed according to the procedure presented in the previous section, is compared against the FSST, SPRT, TSPRT, and 2-SPRT. The comparison is accomplished in terms of the detectability and complexity through computer simulations in zero-mean AWGN environment. More specifically, a large number of i.i.d. noise samples \( \{ z_m \} \) of size \( m \) are generated randomly (and, depending upon the hypothesis, added to the signal) via a C program, so that experiments are performed over a sufficiently large number of iterations. Specifically, we have set the number \( N_I \) of iterations as \( N_I = \frac{10^6}{\beta M} \) when we are dealing with a probability of \( p_0 \): for example, \( 10^{-9} \approx 10^7 \) iterations are executed to get an error probability of \( 10^{-5} \). In addition, we have assumed that \( s_i = 1 \) for \( i = 1, 2, \cdots, m \) so
Table 1  Thresholds of the FSST, SLOT, SPRT, and TSPRT (For TSPRT, $c_1 = c_2 = 0.9$ is used in the simulations).

<table>
<thead>
<tr>
<th></th>
<th>$m$</th>
<th>Upper threshold</th>
<th>Lower threshold</th>
</tr>
</thead>
<tbody>
<tr>
<td>FSST</td>
<td>$m = \frac{\left(Q^{-1}(\alpha_f) - Q^{-1}(1-\beta_f)\right)^2}{\sigma_{H_0}^2}$</td>
<td>$\frac{1}{2}m\theta + \frac{1}{2} \sqrt{m\sigma}(Q^{-1}(\alpha_f) + Q^{-1}(1-\beta_f))$</td>
<td>$\frac{1}{2}m\theta + \frac{1}{2} \sqrt{m\sigma}(Q^{-1}(\alpha_f) + Q^{-1}(1-\beta_f))$</td>
</tr>
<tr>
<td>SPRT</td>
<td>$m \geq 1$</td>
<td>$\frac{1}{2}m\theta + \frac{1}{2} \ln \frac{1-n_0}{n_0}$</td>
<td>$\frac{1}{2}m\theta + \frac{1}{2} \ln \frac{1-n_0}{n_0}$</td>
</tr>
<tr>
<td>TSPRT</td>
<td>$1 \leq m &lt; \frac{\left(Q^{-1}(\alpha_f) - Q^{-1}(1-\beta_f)\right)^2}{\sigma_{H_0}^2}$</td>
<td>$\frac{1}{2}m\theta + \frac{1}{2} \sqrt{m\sigma}(Q^{-1}(\alpha_f) + Q^{-1}(1-\beta_f))$</td>
<td>$\frac{1}{2}m\theta + \frac{1}{2} \sqrt{m\sigma}(Q^{-1}(\alpha_f) + Q^{-1}(1-\beta_f))$</td>
</tr>
<tr>
<td>2-SPRT</td>
<td>$1 \leq m \leq \frac{\left(\frac{1}{\alpha f} \log \frac{1-\alpha(1-\beta_f)}{\beta_f}\right)}{\sigma_{H_0}^2}$</td>
<td>$\frac{1}{2}m\theta + \frac{1}{2} \log \frac{1-\beta_f}{\alpha}$</td>
<td>$\frac{1}{2}m\theta + \frac{1}{2} \log \frac{1-\beta_f}{\alpha}$</td>
</tr>
<tr>
<td>SLOT</td>
<td>$1 \leq m \leq M$</td>
<td>Decided from Eqns. (48)–(51)</td>
<td>Decided from Eqns. (48)–(51)</td>
</tr>
</tbody>
</table>

that we have the joint pdf’s

$$f_{z_i | H_0} = \prod_{i=1}^{m} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{z_i^2}{2\sigma^2}\right)$$

$$= \frac{1}{(2\pi\sigma^2)^{\frac{m}{2}}} \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^{m} z_i^2\right)$$

and

$$f_{z_i | H_1} = \frac{1}{(2\pi\sigma^2)^{\frac{m}{2}}} \cdot \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^{m} (z_i - \theta)^2\right)$$

of observations $z_i$ under the null hypothesis $H_0$ and the alternative hypothesis $H_1$, respectively.

For easy reference, the thresholds of the FSST, SPRT, TSPRT, 2-SPRT, and SLOT are summarized in Table 1. The parameters $c_1$ and $c_2$ for the TSPRT are called the ‘mixing’ coefficients of the FSST and SPRT: it is observed [20] that setting $c_1 = c_2 = 0.9$ produces uniformly smaller ASN for the TSPRT when compared to the FSST without the truncation point being too large.

Normally, we are interested in the case where the actual value of the signal strength is different from the signal strength assumed at the time of the design of detection schemes. In such a case, a useful performance measure representing the detectability is the operating characteristic function (OCF) defined as $1 - P(r)$ in terms of the power function $P(r)$, the probability of accepting the alternative hypothesis [2], where the normalized signal amplitude $r = \frac{\theta}{\sigma}$ denotes the ratio of the actual signal amplitude $\theta$ to the signal amplitude $\theta$ assumed at the time of design. The complexity on the other hand is frequently measured by the number of samples required to make the decision: specifically, we compare the complexity of various schemes depicted in previous sections in terms of the ASN. Needless to mention, it is desirable to detect signals with as low complexity, and at the same time as high detectability, as possible. These are nonetheless conflicting goals in general; as the OCF approaches the ideal of zero, the ASN becomes large, and vice versa.

5.2 Simulation Results

In the simulation results herein, the ratio $\theta^2 / \sigma^2$ of the square $\theta^2$ of signal strength to the variance $\sigma^2$ of the AWGN is set to 1, and all the elements of the signal vector are assumed to be 1. The thresholds of the four schemes SLOT, SPRT, TSPRT, and 2-SPRT are shown in Fig. 4 when $\alpha = \beta = 10^{-3}$; the thresholds $\tau_{LO,\alpha}(m)$ and $\tau_{MF,\beta}(m)$ of the LO-based FSST are also shown for comparison. As explained in the previous section, the upper and lower thresholds of the SLOT cross at around $m = 47$: designed based on the equal error probability at each stage, the threshold of the SLOT is a non-linear function of the stage $m$.

Since Fig. 4 depicts the thresholds under the AWGN
environment in which the test statistic is the sum of observations for all the schemes, the narrower non-decision region (than that of the other schemes) enclosed by the upper and lower thresholds of the SLOT would imply a faster decision, implying a smaller ASN for the SLOT. It should be mentioned, however, that the area does not in general reflect the relative ASN directly since the area depends on the test statistic and the test statistic normally differs from one scheme to another. Interestingly, the 2-SPRT exhibits a triangular continuation region, providing a finite stopping time inherently, which is in a sense similar to the SLOT.

Figure 5 shows the OCF of several detection schemes. It is observed that, for smaller values \( r \) of the normalized signal amplitude (roughly speaking, for \( r < 0.8 \)), the error performance of the SLOT is virtually indistinguishable from that of the TSPRT and 2-SPRT, is better than that of the FSST, and is slightly inferior to that of the SPRT. For larger values of \( r \) (roughly speaking, for \( r > 0.9 \)), on the other hand, the error performance of the SLOT becomes quite close to that of the FSST, and the error performance of the 2-SPRT becomes the best among the schemes compared herein.

Figures 6–8 shows the ASN of the four sequential detection schemes and the FSST. Clearly, as anticipated easily (from Wald-Wolffowitz theorem, for example), the SPRT requires the smallest ASN when \( r = 1 \) (that is, when the actual signal amplitude is the same as the value of the signal amplitude assumed at the time of design) and \( r = 0 \). The complexity of the SPRT is quite low also when \( r > 1 \), which nonetheless becomes quite high around \( r = 0.5 \). The complexities of the 2-SPRT and TSPRT are quite similar to each other: the TSPRT requires higher and lower ASN when \( r \) is near 0.5 and \( r \) is far from 0.5, respectively, compared to the 2-SPRT. It is clearly seen that the SLOT can always detect signals faster than the FSST, TSPRT, and 2-SPRT. In addition, the ASN of the SLOT is smaller than that of the SPRT at most values \( r \) of the normalized signal amplitude, and is considerably smaller than that of the SPRT when the normalized signal amplitude is around 0.5; in essence, the SLOT requires less ASN than the SPRT except when \( r \) is close to 0 and 1.
6. Conclusion
We have first derived a new weak-signal detection scheme called the MF scheme, which employs the LO test statistic in its test function. We have then addressed interesting properties of the LO and MF detection schemes. Specifically, as functions of the sample size, the thresholds of the LO and MF detection schemes have been shown to cross at some value of the sample size; with this fact, we can resolve the problem that the SPRT could in some cases postpone the decision indefinitely.

Based on the characteristics of the thresholds, a novel sequential detection scheme called the SLOT has been proposed. We have compared the complexity of the SLOT with that of the FSST, SPRT, TSPRT, and 2-SPRT in terms of the ASN. It is observed that the SLOT requires always a smaller ASN than the FSST and that the SLOT detects signals faster than the SPRT except when the actual signal amplitude is close to the value assumed at the time of design. In addition, among the sequential schemes settling the problem of indefinite deferral of decision in the SPRT, the complexity of the SLOT has been shown to be slightly lower than that of the TSPRT and 2-SPRT.

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References

Appendix
1. Derivation of Eq. (21): For notational simplicity, let us define
\[ A_i = E \{ g_{LO}(\theta s_i + v_i) \}. \]

Then, we clearly have
\[ \mu_{1,m} = E \left\{ \Lambda_{LO}(\theta s_m + v_m) \right\} \]
\[ = E \left\{ \sum_{i=1}^{m} s_i g_{LO}(\theta s_i + v_i) \right\} \]
\[ = \sum_{i=1}^{m} s_i A_i \]
\[ (A\cdot2) \]

from Eq. (10). Now, based on
\[ \int_{-\infty}^{\infty} g_{LO}(z_i) f_0(z_i) dz_i = -\int_{-\infty}^{\infty} f_0(z_i) dz_i \]
\[ = 0, \]
we can approximate \( A_i \) as
\[ A_i = \int_{-\infty}^{\infty} g_{LO}(z_i) f_0(z_i - \theta s_i) dz_i \]
\[ \approx \int_{-\infty}^{\infty} [g_{LO}(z_i) f_0(z_i) - \theta s_i g_{LO}(z_i) f'_0(z_i)] dz_i \]
\[ = -\theta s_i \int_{-\infty}^{\infty} g_{LO}(z_i) f'_0(z_i) dz_i \]
\[ = \theta s_i I(f_0) \]
\[ (A\cdot4) \]

using Eq. (20) and \( f_0(z_i - \theta s_i) \approx f_0(z_i) - \theta s_i f'_0(z_i) \) for \( \theta \approx 0 \).

Combining Eqs. (A\cdot2) and (A\cdot4), we have
\[ \mu_{1,m} \approx \sum_{i=1}^{m} \theta s_i^2 I(f_0) \]
\[ = 0 \mu_2 I(f_0) p_s(m) \]
\[ (A\cdot5) \]

using Eqs. (3) and (4).

2. Derivation of Eq. (22): For notational simplicity, let us define
\[ B_i = E \{ g_{LO}^2(\theta s_i + v_i) \}. \]

Then, using Eqs. (10) and (A\cdot2), we have
\[ \sigma_{1,m}^2 = E \left\{ \Lambda_{LO}^2(\theta s_m + v_m) \right\} - \mu_{1,m}^2 \]
\[ = \sum_{i=1}^{m} s_i^2 B_i + \sum_{i=1}^{m} \sum_{j=1}^{m} s_i s_j A_i A_j - \sum_{i=1}^{m} \sum_{j=1 \neq i}^{m} s_i s_j A_i A_j \]
\[ = \sum_{i=1}^{m} s_i^2 \left( B_i - A_i^2 \right). \]
\[ (A\cdot7) \]

Now, by noting that
\[ \int_{-\infty}^{\infty} g_{LO}^2(z_i) f_0(z_i) dz_i = 0, \]
\[ (A\cdot8) \]
which is obvious since \( g_{LO}^2(z_i) f_0(z_i) \) is an odd function of \( z_i \), we can obtain the approximation
\[ B_i \approx \int_{-\infty}^{\infty} \left\{ g_{LO}^2(z_i) f_0(z_i) - \theta s_i g_{LO}^2(z_i) f'_0(z_i) \right\} dz_i \]
\[ = \int_{-\infty}^{\infty} g_{LO}^2(z_i) f_0(z_i) dz_i \]
\[ = I(f_0) \]
\[ (A\cdot9) \]

based on \( f_0(z_i - \theta s_i) \approx f_0(z_i) - \theta s_i f'_0(z_i) \) for \( \theta \approx 0 \). Using Eqs. (A\cdot4) and (A\cdot9) in Eq. (A\cdot7), it is easy to obtain
\[ \sigma_{1,m}^2 \approx \sum_{i=1}^{m} s_i^2 \left( I(f_0) - \theta^2 s_i^2 I^2(f_0) \right) \]
\[ \approx \sum_{i=1}^{m} s_i^2 I(f_0) \]
\[ = mI(f_0) p_s(m). \]
\[ (A\cdot10) \]

3. Derivation of Eq. (27): From the fact that the distribution of \( \left( \Lambda_{LO}(\theta s_m + v_m) - \Lambda_{\theta} \right)/\sigma_{\Lambda} \) is asymptotically the standard normal distribution \( N(0,1) \), we have Eq. (26) under the approximations of Eqs. (21) and (22). Then, it is straightforward to obtain Eq. (27) from Eq. (26) after some steps. 

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