Sequential Detection of Signals with Locally Optimum Test Statistic

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ABSTRACT
Based on local optimality, a novel sequential detection scheme is proposed in this paper. The performance of the proposed scheme is compared with that of the sequential probability ratio test (SPRT) and truncated SPRT (TSPRT). The proposed scheme is shown to have higher efficiency, lower complexity, and smaller computational requirements than the SPRT at the cost of partially lower detectability. It is also observed that the proposed scheme has lower complexity and smaller computational requirements than the TSPRT with almost the same level of detectability and efficiency.

1 INTRODUCTION

Based on the generalized Neyman-Pearson lemma of statistical hypothesis testing, a locally optimum (LO) detector maximizes the slope of a power function as the signal-to-noise ratio approaches zero [1]. The class of LO detectors is therefore useful when the strength of signal is much smaller than that of noise processes. In addition, LO detectors generally offer the advantage of simpler structures than uniformly most powerful (UMP) or optimum detectors.

The LO detection has been studied in a variety of environment [2], [3]. Specifically, the LO detection in a generalized observation model is studied in [2]. In [3], the method of LO detection has been applied in multiplicative watermarks.

Along with the full research of the LO detection, the sequential detection has a long history of investigations. As a typical sequential test, the sequential probability ratio test (SPRT) [4] provides significant efficiency in signal detection at moderate accuracy [5]. However, the observation size of the SPRT can be much larger than that of the best fixed sample size (FSS) test as the accuracy increases. To overcome this drawback of the SPRT, the truncated SPRT (TSPRT) has been proposed in [6]. Recently, sequential tests for multihypothesis testing have also been considered in [7].

In this paper, we propose a novel sequential detection scheme based on local optimality, and analyze the performance of the proposed scheme in comparison with the SPRT and TSPRT.

2 OBSERVATION MODEL

In the sequential decision problem, we can postpone the decision at a stage, and test again with more observations at the next stage. At any stage, therefore, this means that the decision space has three elements (or, decisions) \(d_0\), \(d_1\), and \(d_2\), where \(d_0\) and \(d_1\) are the same as in the fixed observation binary decision problem, and \(d_2\) means postponing the decision until the next stage with more data. In essence, the sequential decision problem can be viewed as a ternary decision problem of binary messages.

The discrete-time observation model at the \(m\)th stage can be written as follows:

\[
\begin{align*}
H_{0,m} & : \mathbf{z}_m = \mathbf{v}_m, \\
H_{1,m} & : \mathbf{z}_m = \theta_1 \mathbf{s}_{1,m} + \mathbf{v}_m, \quad \theta_1 > 0.
\end{align*}
\]

(1)

In (1), \(\mathbf{z}_m = [z_{1,m}, z_{2,m}, \ldots, z_{\gamma_m+m}]^T\) is an observation vector, \(\theta_1\) is the signal strength of known value, \(\mathbf{s}_{1,m} = \ldots = \mathbf{s}_{\gamma_m} = \mathbf{0}\) is a null signal vector, and \(\gamma_m = \ldots = \gamma_{\gamma_m} = 1\) is the number of channels.

The decision rule is given by

\[
\begin{align*}
R_0 & : \mathbf{z}_m | H_{0,m}, \\
R_1 & : \mathbf{z}_m | H_{1,m}.
\end{align*}
\]

With the decision rule, the sequential detection is performed as follows:

1. If \(R_0\) is chosen, the decision is made.
2. If \(R_1\) is chosen, the decision is postponed, and the observation is continued for another stage.
3. If \(R_1\) is not chosen, the decision is postponed, and the observation is continued for another stage.

The proposed sequential detection scheme is compared with the SPRT and TSPRT in terms of efficiency, complexity, and computational requirements.
$[v_1, v_2, \ldots, v_{\gamma_n+m}]^T$ is a signal vector, and $v_m = [v_1, v_2, \ldots, v_{\gamma_n+m}]^T$ is the purely additive noise vector of $\gamma_n+m$ independent and identically distributed (i.i.d.) random variables, with $\gamma_n = (m-1)m/2$ representing the accumulated observation size until the $(m-1)^{st}$ stage.

Under the observation model (1), the observation space $Z_m$ (the $\gamma_n+m$ dimensional real space) is partitioned into three decision regions $Z_{i,m}$, $i = 0, 1, 2$ which accept the decision $d_{i,m}$, $i = 0, 1, 2$, respectively.

In this paper, when we know the signal strength $\theta_1$, signal vector $s_{1,m}$, and probability density function (pdf) of the additive noise vector $v_m$, a novel sequential detection scheme is proposed.

3 PROPOSED SCHEME

Consider the sequential test with the two constraints
\[
P\{d_{1,m}|H_{0,m}\} \leq \alpha, \\
P\{d_{0,m}|H_{1,m}\} \leq \beta
\] (2)
for given constants $\alpha$ and $\beta$. Since the sequential decision problem can be viewed as the ternary decision problem of binary messages, there exist six probabilities $P\{d_{i,m}|H_{j,m}\}$, $i = 0, 1, 2$, $j = 0, 1$ satisfying
\[
\sum_{i=0}^{2} P\{d_{i,m}|H_{j,m}\} = 1, \ j = 0, 1.
\] (3)

In the sequential tests, signals in noise are to be detected with minimum observation size and predetermined accuracy. Thus, it is required the probabilities to postpone the decision, i.e., $P\{d_{2,m}|H_{j,m}\}$, $j = 0, 1$ are to be minimized under the constraints of (2) and (3). In other words, $P\{d_{i,m}|H_{j,m}\}$, $i = 0, 1$ are to be maximized under the constraint of (2).

Assuming that two constants $\alpha$ and $\beta$ are small enough to result in three partitioned decision regions [8], the two decision regions $Z_{1,m}$ and $Z_{0,m}$ can be determined individually. Thus, the sequential decision problem can be settled by solving the problems
\[
\max P\{d_{0,m}|H_{0,m}\} \text{ subject to } P\{d_{0,m}|H_{1,m}\} \leq \beta
\] (4)
and
\[
\max P\{d_{1,m}|H_{1,m}\} \text{ subject to } P\{d_{1,m}|H_{0,m}\} \leq \alpha
\] (5)

After some manipulations using Lagrange multipliers and applying the method of LO detection to the test statistic, the three decision regions can be written as follows:
\[
\begin{align*}
Z_{0,m} : \ & \{z_m|\Lambda_{LO}(m) \leq B(m)\}, \\
Z_{1,m} : \ & \{z_m|\Lambda_{LO}(m) \geq A(m)\}, \\
Z_{2,m} : \ & \{z_m|B(m) < A_{LO}(m) < A(m)\}.
\end{align*}
\] (6)

In (6),
\[
\Lambda_{LO}(m) = \frac{f^{(v)}(z_m|H_{1,m})|_{\theta_1=0}}{f(z_m|H_{0,m})}
\] (7)
is the test statistic at the $m^{th}$ stage, $v$ is the first non-zero derivative at $\theta_1 = 0$, and $A(m)$ and $B(m)$ are two thresholds at the $m^{th}$ stage which are obtained to satisfy
\[
P\{\Lambda_{LO}(m) \geq A(m)|H_{0,m}\} = \alpha, \\
P\{\Lambda_{LO}(m) \leq B(m)|H_{1,m}\} = \beta.
\] (8)
Here, $f(z_m|H_{0,m})$ and $f(z_m|H_{1,m})$ are the pdf of the observation vector $z_m$ given that the null and alternative hypotheses $H_{0,m}$ and $H_{1,m}$ are true, respectively.

In (6), we can observe that $m$ new observations (in addition to the $\gamma_n$ previous ones) are used to select a decision region at the $m^{th}$ stage. Thus, as the number of stages increases, the total observation size could increase dramatically. This problem can be resolved by the truncation strategy [6]. Let us define the number $\kappa$ as
\[
\kappa = \arg \max_n \left\{ \frac{n(n+1)}{2} < \delta \right\},
\] (9)
where $\delta$ is the observation size required for the best FSS test.

Then, the procedure of the proposed scheme for $1 \leq m \leq \kappa$ can be written as follows:
\[
\begin{align*}
\Lambda_{LO}(m) & \leq B(m), & \text{Accept } H_{0,m}. \\
\Lambda_{LO}(m) & \geq A(m), & \text{Accept } H_{1,m}. \\
B(m) & < \Lambda_{LO}(m) < A(m), & \text{Test again with additional } m+1 \text{ observations.}
\end{align*}
\] (10)

When $m = \kappa + 1$, the procedure of the proposed scheme can be written as follows:
\[
\Lambda_{LO,T}(\delta) < C_T(\delta), & \text{ Accept } H_{0,T}. \\
\Lambda_{LO,T}(\delta) \geq C_T(\delta), & \text{ Accept } H_{1,T}.
\] (11)

In (11),
\[
\Lambda_{LO,T}(\delta) = \frac{f^{(v)}(z_{\delta,T}|H_{1,T})|_{\theta_1=0}}{f(z_{\delta,T}|H_{0,T})}
\] (12)
is the test statistic at the truncation stage,
\[
z_{\delta,T} = [z_1, z_2, \ldots, z_{\delta}]^T
\] (13)
is the observation vector at the truncation stage, $C_T(\delta)$ is the threshold used for the best FSS test, and $H_{0,T}$ and $H_{1,T}$ are the null and alternative hypotheses at the truncation stage.

4 PERFORMANCE ANALYSIS AND SIMULATION RESULTS

4.1 PERFORMANCE ANALYSIS

The test statistic of a detector determines the structure of the detector. Based on local optimality, the test statistic of the proposed scheme is less than, or at most as complicated as, that of the SPRT and TSPRT. In addition, the test statistic of the proposed scheme does not depend on the signal strength $\theta_1$. Thus, we are not required to have prior knowledge of
to that for the SPR T and TSPR T can be approximated both as slightly higher ASN than the SPR T near the known signal vector OCFs and the ASN functions of the SPR T and the TSPR T are lower than that of the SPR T. In comparison with the TSPR T, the difference of the ASN between the proposed scheme and the TSPR T, the mixture constants of the scheme, FSS test, SPR T, and TSPR T are partially lower than that of the SPR T. In comparison with the TSPR T, the proposed scheme requires less stages for the same observation size, and has lower complexity in the structure.

4.2 SIMULATION RESULTS

In the performance measurements of sequential detection schemes, the operating characteristic function (OCF) and average sample number (ASN) are directly related to detectability and efficiency [4], respectively. For an easy and distinct comparison, simulations of the four detection schemes are performed under the zero-mean, unit variance purely additive white Gaussian noise (AWGN). The ratio of the signal strength $\theta_1$ to a standard deviation $\sigma$ is set to 1, and the known signal vector $s_i, m$ is fixed to $[1, 1, \ldots, 1]^T$. The OCFs and the ASN functions of the SPR T and TSPR T are simulated based on [5] and [6], respectively. Specifically, in the TSPR T, the mixture constants of 0.5 are selected to satisfy the two predetermined error probabilities and maximize the efficiency via simulation.

As shown in Fig 1, detectability of the proposed scheme is partially lower than that of the SPR T. In comparison with the TSPR T, the difference of detectability between the proposed scheme and the SPR T is unnoticeable.

Fig 2 shows the ASN functions of the four detection schemes when $\alpha = \beta = 10^{-3}$. The proposed scheme has slightly higher ASN than the SPR T near $\tau = \theta_1/\theta_1 = 0$ and 1. At $\tau = 0.5$, the ASN of the proposed scheme is noticeably lower than that of the SPR T. In comparison with the TSPR T, the difference of the ASN between the proposed scheme and the TSPR T is negligible.

5 CONCLUDING REMARK

In this paper, a novel sequential detection scheme based on local optimality is proposed for the discrete-time observation model. The proposed scheme has higher efficiency, lower complexity, and smaller computational requirements at the cost of partially lower detectability than the SPR T at the same size and power. Since the computational requirements and the observation size of the SPR T become unacceptable as the two error probabilities become smaller, the proposed scheme is more favorable in this sense. In comparison with the TSPR T, the proposed scheme requires less complexity and smaller computations than the TSPR T with almost the same level of detectability and efficiency.

REFERENCES


