Asymptotic Performance of Weak Signal Detectors in Multiplicative and First-Order Markov Additive Noise Environment

December 10, 2005

Statistical Signal Processing Laboratory

Department of Electrical Engineering and Computer Science
Korea Advanced Institute of Science and Technology
Contents

- Introduction
- The observation model
- Locally optimum test statistics
- Performance analysis and comparison
- Conclusion
Introduction (I)

- **The independent noise assumption**
  - In most signal detection problem, it is generally assumed that the sampled noise components are statistically independent.
  - The independent noise assumption is frequently violated in discrete-time signal detection applications.

- **The first-order Markov model**
  - It has been reported that many signals and noise in telecommunication systems are of this type.
Introduction (II)

- The locally optimum (LO) detector
  - With the recent increasing interest in developing low power communication systems, the importance of weak signal detection keeps growing steadily.

- LO detection under first-order Markov noise environment
  - Based on the rationale of the necessity of dependent observation model and LO detection, LO detection of known signals is considered under the multiplicative and first-order Markov additive noise.
The observation model (I)

- **The discrete-time observation model**

\[ X_i = \theta e_i M_i + W_i, \quad i = 1, 2, \ldots, n \]

- \( \{X_i\} \): the discrete-time observations
- \( \{M_i\} \): multiplicative noise components
- \( \{W_i\} \): additive noise components
- \( \{e_i\} \): the known signals
- \( \theta \): the signal strength
- \( n \): the number of observations in a sample
The observation model (II)

The probability density functions (pdfs)

- $f_{W_i}()$, $f_{M_i}()$: the pdfs of $W_i$ and $M_i$, respectively

- $f_W(w)$, $f_M(m)$: the joint pdfs of $W=(W_1, W_2, \ldots, W_n)$ and $M=(M_1, M_2, \ldots, M_n)$, respectively, where $w=(w_1, w_2, \ldots, w_n)$ and $m=(m_1, m_2, \ldots, m_n)$
The observation model (III)

- $f_{\tilde{W}_i}(W_i | W_{i-1}) = f_i(W_i | W_{i-1})$
  
  the transition or conditional pdf of $W_i$ given $W_{i-1}$ where
  
  $\tilde{W}_1 = W_1$ and $f_{\tilde{W}_1}(W_1 | W_0) = f_1(W_1)$

- **Assumption**
  
  The pdfs $f_M$ and $\{f_i\}$ are smooth enough and satisfy regularity conditions.
The observation model (IV)

- A hypothesis testing problem

\[ H : \quad \phi_X(x \mid \theta), \quad \theta = 0 \]
\[ K : \quad \phi_X(x \mid \theta), \quad \theta > 0 \]

where

\[
\phi_X(x \mid \theta) = \int_{R^n} f_M(m) \left\{ \prod_{i=1}^{n} f_i(y_i(\theta) \mid y_{i-1}(\theta)) \right\} dm,
\]

\[
y_i(\theta) = x_i - \theta e_i m_i, \text{ and } dm = dm_1 dm_2 \ldots dm_n
\]
Locally optimum test statistics

- The LO test statistic for $\theta > 0$ versus $\theta = 0$

$$T_{LO}(x) = \left. \frac{\partial^\nu \phi_X(x \mid \theta)}{\partial \theta^\nu} \right|_{\theta = 0} \frac{\phi_X(x \mid \theta)}{\phi_X(x \mid 0)}$$

- $\nu$: the smallest natural number for which (1) does not vanish.
Locally optimum test statistics

When the means of multiplicative noise components are not all zero (I)

- The LO test statistic

\[ T_{LO_0}(x) = \sum_{i=1}^{n} \alpha_i g_{A,i}(x_{i-1}, x_i, x_{i+1}) \]

\[ x_0 = x_{n+1} = 0, \quad e_0 = e_{n+1} = 0, \quad \alpha_i = e_i E\{M_i\} \]

\[ g_{A,i}(x, y, z) = \tilde{g}_{10,i}(y \mid x) + \tilde{g}_{01,i+1}(z \mid y), \quad i = 1, 2, \ldots, n \]

\[ \tilde{g}_{pq,i}(x \mid y) = \begin{cases} \frac{\nabla_x^p \nabla_y^q f_i(x \mid y)}{f_i(x \mid y)}, & i = 1, 2, \ldots, n \\ 0, & i \leq 0 \quad \text{or} \quad i \geq n + 1 \end{cases} \]

\[ \nabla_a = -\frac{\partial}{\partial a} \]
Locally optimum test statistics
When the means of multiplicative noise components are not all zero (II)

The LO test statistic when the additive noise components are independent

\[ T_{LO1}(x) = \sum_{i=1}^{n} \alpha_i g_{1,i}(x_i) \]

\[ g_{1,i}(x) = -\frac{f'_{W_i}(x)}{f_{W_i}(x)} \]

the nonlinearity defined in previous works for the independent and identically distributed (i.i.d.) noise.
Locally optimum test statistics

When the means of multiplicative noise components are all zero (I)

- The LO test statistic

\[
T_{LO2}(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k} K_M(i, j) g_{A,i} (x_{i-1}, x_{i}, x_{i+1}) g_{A,j} (x_{j-1}, x_{j}, x_{j+1}) \\
+ \sum_{i=1}^{n} K_M(i, i) h_{A,i} (x_{i-1}, x_{i}, x_{i+1}) + \sum_{i=1}^{n} K_M(i, i + 1) h_{D,i} (x_{i}, x_{i+1})
\]

- \( h_{A,i} (x, y, z) = \tilde{g}_{20,i} (y \mid x) - \tilde{g}_{10,i}^{2} (z \mid y) \) \\
- \( h_{D,i} (x, y) = \tilde{g}_{11,i+1} (y \mid x) - \tilde{g}_{10,i+1} (y \mid x) \tilde{g}_{01,i+1} (y \mid x), \quad i = 1, 2, \ldots, n \) \\
- \( K_M(i, j) = e_i e_j E \left\{ M_i M_j \right\} \)
Locally optimum test statistics

When the means of multiplicative noise components are all zero (II)

- The test statistic when the additive noise components are an independent random process

\[ T_{LO3}(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} K_M(i, j) g_{1,i}(x_i) g_{1,j}(x_j) + \sum_{i=1}^{n} K_M(i, i) \left\{ g_{2,i}(x_i) - g_{1,i}^2(x_i) \right\} \]

since \( \tilde{g}_{pq,i}(x \mid y) = 0 \) for \( q \geq 1 \), \( \tilde{g}_{10,i}(x \mid y) = g_{1,i}(x) \), where

\[ g_{2,i}(x) = \frac{f_{W_i}''(x)}{f_{W_i}(x)}. \]
Locally optimum test statistics
When the means of multiplicative noise components are all zero (III)

- The test statistic when the multiplicative noise components are an independent random process

\[
T_{LO4}(x) = \sum_{i=1}^{n} e_i^2 \sigma_{M,i}^2 \left\{ \tilde{g}_{20,i}(x_i | x_{i-1}) + 2 \tilde{g}_{10,i}(x_i | x_{i-1}) \tilde{g}_{01,i+1}(x_{i+1} | x_i) + \tilde{g}_{02,i+1}(x_{i+1} | x_i) \right\}
\]

\[- \sigma_{M,i}^2 = E\left\{ M_i^2 \right\}

the variance of the multiplicative noise \( M_i \).\]
Locally optimum test statistics

When the means of multiplicative noise components are all zero (IV)

- The test statistic when the additive and multiplicative noise components are independent random processes

\[ T_{LO5}(x) = \sum_{i=1}^{n} K_M(i,i)g_{2,i}(x_i) \]
Performance analysis and comparison

Noise distribution (I)

- The statistical characteristics of the multiplicative and additive noise components
  - The LO test statistics depend only on the first- and second-order characteristics of \( \{M_i\} \).
  
  \[
  E\{M_iM_j\} = r_{M|i-j|} + E\{M_i\}E\{M_j\}, \quad |r_M| < 1
  \]
Performance analysis and comparison

Noise distribution (II)

- To examine the influence of the statistical characteristics of the additive noise components, we should consider distribution of \( \{W_i\} \).

\[
f_{Y|X}(y \mid x) = \frac{\exp\left\{-\left[y - m_y - r \frac{\sigma_y}{\sigma_x} (x - m_x)\right]^2 / 2\sigma_y^2 (1 - r^2)\right\}}{\sqrt{2\pi\sigma_y^2 (1 - r^2)}}
\]
Performance analysis and comparison

Performance measure

- Asymptotic Performance
  - The asymptotic relative efficiency (ARE)

\[
\text{ARE}_{1,2} = \frac{\xi_1}{\xi_2},
\]

where

\[
\xi_i = \lim_{{n \to \infty}} \frac{\left( \frac{d^\nu E_1 \{T_{i,n}(X)\}}{d\theta^\nu} \right)_{\theta=0} \left( \frac{nV_0 \{T_{i,n}(X)\}}{\phi_X(x|\theta)} \right)_{\theta=0}}{nV_0 \{T_{i,n}(X)\}},
\]

and

\[
\frac{d^\nu E_1 \{T_{i,n}(X)\}}{d\theta^\nu} \bigg|_{\theta=0} = \int_{R^n} T_{i,n}(X) \frac{d^\nu \phi_X(x|\theta)}{d\theta^\nu} \bigg|_{\theta=0} dx.
\]
Performance analysis and comparison
When the means of multiplicative noise components are not all zero

- **Asymptotic performance**: \( \text{ARE}_{LO0,LO1} \)

![Graph showing \( \text{ARE}_{LO0,LO1} \) under the FMG environment with various \( \alpha \) values.](image-url)
Performance analysis and comparison

When the means of multiplicative noise components are all zero (I)

- **Asymptotic performance:** $\text{ARE}_{\text{LO2, LO3}}$

\[ \text{ARE}_{\text{LO2, LO3}} \text{ under the FMG environment} \]
Performance analysis and comparison

When the means of multiplicative noise components are all zero (II)

- **Asymptotic performance:** $\text{ARE}_{L02, L04}$

![Graph showing $\text{ARE}_{L02, L04}$ under the FMG environment](image)

- Statistical Signal Processing Laboratory
Performance analysis and comparison
When the means of multiplicative noise components are all zero (III)

- **Asymptotic performance:** $\text{ARE}_{LO2, LO5}$

![Graph showing performance analysis](image)
Conclusion

- We have derived detection test statistics for weak known signals in observations corrupted by multiplicative and first-order Markov additive noise.
- The derived test statistics depend on the statistical characteristics of \( \{M_i\} \) and \( \{W_i\} \).
Conclusion

- To see the performance of several detectors clearly, we have obtained and compared the asymptotic performance of detectors under multiplicative and first-order Markov additive noise environment.